HINTS ON DUAL VARIABLES FROM THE LATTICE SU(2) GLUODYNAMICS

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In many cases, topological excitations become fundamental variables in a dual formulation of the theory. Assuming this is true in case of $SU(2)$ gluodynamics, we look for hints on the dual variables from the lattice simulations. Two-dimensional vortices, or branes with recently established properties seem to be most natural candidates. The total area of the branes scales in the physical units while the non-Abelian action is ultraviolet divergent. The branes are populated with magnetic monopoles, or tachyonic mode.

1. Introduction

Duality proves a powerful tool to get insight into dynamics of a system, for a review see, e.g., [1]. A recent landmark is a proof of the confinement using duality within certain supersymmetric Abelian field theories [2]. There exist also other examples of Abelian gauge theories which allow for dual formulations in terms of electric and magnetic variables [1]. As for non-Abelian theories, it has become clear that search for dualities goes in this case beyond field theories, see, in particular, [3]. This makes the search for dualities even a more profound problem.

Search for dual formulations is a theoretical challenge, first of all. Here we focus on a much less ambitious aspect of the problem, trying to understand what kind of hints on the dual variables can be extracted from the lattice simulations of $SU(2)$ gluodynamics. The point is that topological excitations found within the original formulation can become dual variables, see, e.g., [1]. Let us explain this on the Seiberg-Witten example. In $N = 2$ SUSY Abelian theory there exist classical monopole solutions with mass equal to

$$M_{mon} \;=\; \langle\phi\rangle \cdot Q_M \quad , \tag{1}$$

where ϕ is a scalar filed and Q_M is the magnetic charge which is inverse proportional to the electric charge, $Q_M = const/e$. Moreover, the solution saturates the BPS bound. The $N = 2$ supersymmetry ensures absence of quantum corrections to the mass. Thus, one can prove that the monopole is becoming light for large original couplings e. This observation can be considered as a hint that there exists a dual formulation in terms of the monopoles.

In case of gluodynamics, effective degrees of freedom responsible for the confinement are natural candidates for future dual variables. Phenomenologically, magnetic monopoles and P-vortices are most promising, for a recent review see [4]. Note that both monopoles and P-vortices condense which is a common sign of the dual variables.

In this paper we concentrate on field theoretical properties of the monopoles and vortices. Trying to appreciate the anatomy of the monopoles in terms of the Yang-Mills fields, we note first that apriori it would be easy to argue that the monopoles might not play any dynamical role. Indeed, there are two known types of Abelian monopoles, corresponding to singular and non-singular fields at the origin. We argue that generically both types of the Abelian monopoles cannot be generalized to the non-Abelian case. The only possibility left is a self tuning of the suppression due to the action and of enhancement due to the entropy. Both should be ultraviolet divergent but cancel each other to the order Λ_{QCD}.

There is no known mechanism for the self-tuning in the original Yang-Mills theory and the possibility looks pure academic. Amusingly enough, lattice data do provide strong evidence for such self-tuning. Theoretically, one can then go further and argue that self-tuning of the monopoles is not sufficient, even if it is granted by the data, and monopoles should belong to self-tuned two-dimensional surfaces. The data turn again to support this conclusion and the surfaces in point are the P-vortices, for review see [4]. The P-vortices are indeed closely associated with monopoles. What makes the self-tuning manifest is a recent observation that the P-vortices and monopoles have non-Abelian action which is divergent in the ultraviolet [5,6,7]. In other words, the lattice spacing a is seen in distribution of the non-Abelian action on and around the vortices. On the other hand, Λ_{QCD} controls the monopole mass and vortex string tension.

We will call these thin two-dimensional surfaces populated with tachyonic mode (monopoles) branes. (The term vortex is somehow reserved, in the studies of confinement, for thick, or bulky objects, see, e.g., [4,8].) In Sect. 2 we discuss self-tuning of the magnetic monopoles in the lattice

$SU(2)$. In Sect. 3 we describe evidence in favor of self-tuning of the vortices. Conclusions are in Sect. 4. Note that the argumentation outlined above was already presented partly in Refs. [9,10].

2. Self-tuning of the lattice monopoles

2.1. *Fine tuning of the Abelian monopoles*

The size of the BPS monopole (1) is large, that is of order $\langle\phi\rangle^{-1}$. The softening of the fields is due to a non-trivial scalar field configuration. Now we turn to another well understood case of the monopole condensation, that is compact $U(1)$, for review see, e.g., [11]. Generically, the idea is the same as above: condensation of the monopoles is signaled by vanishing of the monopole mass. There are two important changes, however. First, there is no scalar field so that magnetic field is very strong at short distances. Second, lattice theories are considered in the Euclidean space.

In more detail, we consider electromagnetic field on the lattice. In the continuum limit the Lagrangian is that of free field:

$$L ~=~ \frac{1}{4e^2}F_{\mu\nu}F_{\mu\nu} ~~. \tag{2}$$

where $F_{\mu\nu}$ is the field strength tensor. Since there is no softening Higgs field the radiative mass of the monopole is divergent in the ultraviolet:

$$M_{rad} ~=~ \frac{1}{8\pi}\int_a^{\infty} d^3r|\mathbf{H}|^2 ~\approx~ \frac{const}{e^2}\frac{1}{a} ~~, \tag{3}$$

where a is an ultraviolet cut off, $\mathbf{H}$ is the magnetic field, $|\mathbf{H}| \sim 1/(er^2)$ and e is the electric charge appearing because of the Dirac quantization condition.

Because of the ultraviolet divergence (3), the monopoles can be considered point-like. The probability to observe a monopole trajectory of the length L is suppressed then by the action as

$$exp(~-S) ~\sim~ \exp\big(~-\frac{const}{e^2}\frac{L}{a}\big) ~~, \tag{4}$$

and at first sight the monopoles are removed from the physical spectrum in the limit $a \to 0$. Remarkably, this conclusion is actually not true. Indeed, the probability $W(L)$ to observe a trajectory of length L is a product of (4) and the entropy factor N_L which is the number of ways the length L can be realized on the lattice. The latter is also ultraviolet divergent and:

$$W(L) ~\sim~ \exp(-M_{ren}\cdot L) ~~, \tag{5}$$

where the renormalized mass is approximately:

$$M_{ren} \;\approx\; \frac{1}{a}\Big(\frac{const}{e^2} - \ln 7\Big) \quad , \tag{6}$$

where the $\ln 7$ term is of pure geometrical origin, due to the entropy.

Monopole condensation occurs when any trajectory length is not suppressed by (5). In other words, choosing $e^2 = e^2_{crit}$, where

$$\frac{const}{e^2_{crit}} - \ln 7 \;=\; 0 \quad , \tag{7}$$

ensures condensation of the monopoles. Validity of this condition was checked numerically, see [12] and references therein.

More generally, starting with the classical action for a point-like particle:

$$S_{cl} \;=\; M(a)\cdot L \quad , \tag{8}$$

one can develop field theory in the Euclidean space-time in the polymer representation, see, e.g. [13]. In particular, evaluating the propagator as a path integral a la Feynman one arrives at the following expression for the propagating, or physical mass:

$$m^2_{phys} \;\approx\; \frac{8}{a}\big(M(a) - \ln 7/a\big) \quad , \tag{9}$$

where the factor $\sim \ln 7$ is specific for the hyper-cubic lattice in $d = 4$ and for charged particles (closed trajectories).

In other words, the condensation occurs again at the point where the monopole mass vanishes, $m^2_{phys} = 0$. Moreover, the mass can be fine tuned to zero by choosing the corresponding value of the electric charge (7).

2.2. *No-go arguments for the $SU(2)$ case*

Imagine now that we would try to extract lessons from the Abelian cases and suggest a mechanism for having a light scalar particle (monopole) in the non-Abelian case. We would fail completely and come actually to a kind of a no-go theorem for the non-Abelian case.

Mild fields. It is reasonable to consider separately mild and hard gluonic fields. By mild fields we understand an analog of the Polyakov- 't Hooft monopole while by hard fields we understand an analog of the monopoles of the compact (lattice) $U(1)$.

There is no Higgs field in our case. Monopoles can still be defined following Ref. [14]. Namely, pick up any scalar field in the triplet representation, H^a. Then vanishing of this field,

$$H^a \;=\; 0 \;,\; (a = 1, 2, 3) \tag{10}$$

represent three conditions specifying a world line in $d = 4$ space. The central point is that there exists such a definition of the monopole charge that this trajectory is nothing else but the monopole trajectory [14].

The problem is to identify, in absence of matter, a suitable H^a field. Indeed, locally the field H^a is to reduce to a product of non-Abelian field strength tensors $G^a_{\mu\nu}$. The simplest possibility is to choose the Higgs field proportional to a particular component of the field strength tensor, say,

$$H^a \; = \; G^a_{12} \; .$$

This choice would violate Lorentz invariance, however. If we try a Lorentz scalar then the simplest combination vanishes identically:

$$\epsilon^{abc} G^b_{\mu\nu} G^c_{\mu\nu} \; \equiv \; 0 \; .$$

Increasing the number of the $G's$ in the product makes, first of all, the construction unrealistically complicated. Moreover, a more detailed consideration shows that it does not help at all [15].

Thus, there is no way to construct a light monopole in terms of mild, i.e. nonsingular fields.

Hard fields. Try now to construct a fine tuned monopole, in analogy with the lattice $U(1)$ theory. In the $U(1)$ case the price paid for the absence of the Higgs field is singularities of the gauge field. In particular, the magnetic current is defined in terms of violations of the Bianchi identities,

$$j^{mon}_{\mu} \; \sim \; \epsilon_{\mu\nu\rho\sigma} \partial_{\nu} F_{\rho\sigma} \; . \tag{11}$$

Violation of the Bianchi identities assumes the fields to be singular. However, all the expressions are regularized and well defined on the lattice [16].

In the $SU(2)$ case definition (11) can be generalized to a covariant form in an obvious way. Violation of the Bianchi identities means that the non-Abelian fields are singular in the continuum limit $a \to 0$. Singular fields imply a singular action. Singular action means that the corresponding fluctuation is removed from the physical spectrum unless the radiative mass is fine tuned to the entropy factor. So far, everything is similar to the $U(1)$ pure gauge theory. At this point, the analogy breaks down, however. A non-Abelian theory is an interacting theory even in the absence of matter. As a result the coupling is running and cannot be used as a parameter to tune the propagating mass (9).

Thus, we come to the conclusion that neither finite nor singular non-Abelian fields can ensure appearance of a light scalar particle (monopole).

2.3. *Lattice monopoles*

The no-go arguments above have been realized rather recently. Moreover, the phenomenological studies of the lattice monopoles flourish, for a review see, e.g., [17]. Now, however, we can look backwards and ask, why the no-go arguments did not work. This question can be answered quantitatively, by measuring directly the entropy and *non-Abelian* action of the monopoles (we always mean monopoles of the maximal Abelian projection, see, e.g., [17]). For the monopole action the result of measurements is [5]:

$$S_{mon} \;\approx\; \ln 7 \cdot \frac{L}{a} \;, \tag{12}$$

and for the entropy one gets [18]:

$$N_L \;\approx\; (L/a)^7 \;\; . \tag{13}$$

We do not discuss here error bars in Eqs (12), (13): details can be found in the original papers. However, the answer to our question is quite clear, compare (12), (13) and (9). Namely, the monopoles exist because of a self-tuning. Both the action and entropy are ultraviolet divergent but cancel each other so that the propagating monopole mass is controlled by Λ^{-1}_{QCD} [10]. Note that the action is tuned to a geometrical factor. On the other hand, the action of a pure Abelian monopole (embedded into the $SU(2)$) would be proportional to $g^{-2}(a)$ and could not be tuned to $\ln 7$ for varying a. The mechanism of the self-tuning in terms of the original Yang-Mills fields is not known. However, it follows from the arguments above that this self-tuning was the only "chance" for the monopoles to exist.

The self-tuning of the monopoles can be tested further, through studies of properties of the monopole clusters. In particular, one can predict the spectrum of finite clusters, $P(L)$ and their radii as functions of their length [19]:

$$P(L) \;\sim\; L^{-3} \;, \;\; R(L) \;\sim\; \sqrt{L\cdot a}\;. \tag{14}$$

The prediction is known to be true on the lattice [20,21].

The main idea behind (14) is to use relations common to theory of percolating systems. For further applications of the percolation theory to monopoles see [21,22]. Physicswise, what is most intriguing about the success of the percolation theory is that it treats monopoles as point like at the scale of a. Thus, the monopoles emerge as a new probe of short distances.

3. Self-tuning of P-vortices

3.1. *No-go arguments, second round*

Although the self-tuning of the monopoles allows to describe a lot of data on the monopole clusters [19,21,22], it brings us to a new question which looks even more devastating than the original puzzles. Namely, if relations (12) and (13) is the ultimate truth about the lattice monopoles, then according to (9) we have a light scalar particle. But because of the asymptotic freedom we may have only three original gluons at short distances.

To scrutinize the argument we should translate the problem into the language of the observables, that is monopole trajectories. Note, first of all, that point-like particles in field theory are counted through ultraviolet divergences. Usually, in the continuum theory one deals with logarithmic divergences. In particular, the fact that there are only gluons at short distances can be checked by studying the running of the coupling.

However, the logarithmic running is a relatively small effect, difficult to resolve. Luckily enough, on the lattice one can measure also power-like divergences since the ultraviolet regularization is known explicitly. In particular, one can count particles by measuring the average plaquette action. The standard prediction is:

$$\langle (G^a_{\mu\nu})^2 \rangle \;\approx\; \frac{(N_c^2-1)}{a^4} \tag{15}$$

Indeed, in the limit $a \to 0$ the plaquette action is to count zero-point fluctuations of gluons. Prediction (15) is easy to check. Moreover, the relation (15) has perturbative corrections calculable, again, in terms of gluons alone. In fact, many perturbative terms are explicitly known [23].

The first non-perturbative contribution to the plaquette action which is allowed by the continuum theory is that of the ultraviolet renormalon, for a recent review and further references see [24]. This observation severely constrains the monopole contribution to the plaquette action:

$$\langle (G^a_{\mu\nu})^2 \rangle_{mon} \;\;\sim\;\; \Lambda^2_{QCD}\cdot a^{-2}\;. \tag{16}$$

To appreciate implications of (16) for the monopole trajectories we need to recollect some elements of the geometrical picture. Monopoles occupy centers of elementary cubes on the lattice. The cubes add up to trajectories. The action, associated with the monopole is measured as excess of action density on the plaquettes forming the cubes times the volume of the cube, a^3. In particular, (12) means that the excess of the action density on the

plaquettes belonging to the monopoles is comparable to action density of the zero-point fluctuations (15).

Now, for the purpose of orientation consider the standard model of percolation, see, e.g., [25]. One introduces probability p for a cube to belong to a monopole trajectory. At some value $p = p_{cr}$ the trajectories begin to percolate, i.e. there appears a single cluster which spreads over the whole lattice. Such a phenomenon is observed for the lattice monopoles indeed. However, in this model, the total number of monopole cubes would be a finite fraction, p of all the cubes on the lattice. As a result, the excess of the action due to the monopoles, averaged over the lattice would be comparable to (15). and the monopoles would interfere with the counting degrees of freedom. We uncover again the danger that observations (12), (13) signify presence of new particles at short distances, in contradiction with the asymptotic freedom.

Thus, ordinary percolation is not allowed for the monopoles. And what is allowed? Introduce probability $\theta(cube)$ for an arbitrary cube on the lattice to contain a monopole. Then (16) implies:

$$\theta\ (cube)\ \sim\ \left(a\cdot\Lambda_{QCD}\right)^2\ . \tag{17}$$

That is, the probability for a cube to contain a monopole with ultraviolet divergent action should itself depend on a. The prediction might look even absurd at first sight but the drama is that the data at presently available lattices [21,26] do comply with (17).

3.2. *Branes*

Although observation of (17) on the lattice settles our problem with asymptotic freedom, the overall picture might look still very puzzling. Namely, the lattice monopoles share the properties (12), (13) with a free particle in $d = 4$ space. On the other hand, the probability (17) depends on the lattice spacing a, in striking distinction from the free particle in $d = 4$. The geometrical meaning of (17) is actually transparent: monopoles live on a $d = 2$ subspace of the whole $d = 4$ space. Indeed, the factor $(a \cdot \Lambda_{QCD})^2$ gives then the weight of the cubes which belong to the surface provided that the area of the surface is in physical units.

The conclusion might sound bizarre but actually phenomenologically this picture, at least partially, has been known. Namely, the surfaces are nothing else but the P-vortices, for review see [4]. Their total area scales in

physical units [27,4]. According to the latest data [6]:

$$A_{vort} \approx 24\ (fm)^{-2} \cdot V_4\ , \tag{18}$$

where V_4 is the lattice volume. Moreover it is known from the numerical simulations that monopoles are strongly correlated with the P-vortices. This was observed first for a single value of the lattice spacing a [28] and later confirmed for all the values of a available [7].

Although scaling (18) and conspiracy of the P-vortices and monopoles were known since some time, no self-tuning was discussed until very recently [6]. The missing point was the measurement of the vortex action as function of the lattice spacing a. The (non-Abelian) action turned to be ultraviolet divergent [6]:

$$S_{vort} \approx 0.54\frac{A}{a^2}\ . \tag{19}$$

Observation (19) means that the self-tuning is generalized from the monopoles to P-vortices [6]. Moreover, conspiracy of the monopoles and P-vortices is elevated from a simple phenomenological observation to a consequence of the asymptotic freedom.

3.3. *Interplay of the monopoles and vortices*

Let us emphasize that it would be an oversimplification to say that the vortices are primary and monopoles are secondary. The interplay between the monopoles and vortices is subtler. Namely, at short distances monopoles 'know' about $d = 4$. Indeed, the spectrum of finite clusters (14) is sensitive to the number of dimensions:

$$N(L) \sim L^{-1-d/2}\ , \tag{20}$$

and the data indicate $d = 4$, [19,21]. Also, it is the monopole action which is tuned to the entropy of a point-like particle in $d = 4$, compare (12) and (13). Note that both the action and entropy are decided at short distances. There is no such match for the vortices if one would assume that the vortices are structureless [6].

On the other hand, in the infrared the monopoles are associated with a $d = 2$ subspace of the full $d = 4$ space. Thus, it is condensation of the branes that is observed, not of monopoles.

4. Conclusions

The lattice data indicate that there exist branes in the vacuum state of the lattice $SU(2)$ gluodynamics which possess remarkable properties:

(a) total area scales in physical units;
(b) non-Abelian action is ultraviolet divergent;
(c) branes are populated with scalar particles, monopoles;
(d) monopoles are particles living on a $d = 2$ subspace;
(e) the branes percolate and condense.

The branes are natural candidates for dual variables.

An important caveat is that in all the cases the evidence is entirely numerical and is, therefore, true for a limited range of the lattice spacings available now. The only theoretical support for the picture observed we could find is that if existence of a dual description were granted then the basic features of the branes listed above would be uniquely fixed.

Appearance of the branes might be related to a very special status of dimension two condensates in gauge theories , see, in particular, [29]. Naively, the first gauge-invariant condensate, $<(G^2_{\mu\nu}>$ has dimension four. The only Lorentz invariant operator of dimension two, $(A^a_\mu)^2$ is seemingly gauge dependent. However, one can argue that the minimal value of $<(A^a_\mu)^2>$, over the gauge orbit is gauge invariant. This minimization condition implies non-locality. In case of pure electromagnetic field

$$< A^2_\mu >_{min} \;=\; (const)\int \frac{d^4xd^4x^{'}}{(x-x^{'})^2}F_{\mu\nu}(x)F_{\mu\nu}(x^{'}) \quad . \tag{21}$$

In the non-Abelian case there is no such closed relation but an explicit calculation in the Hamiltonian formalizm [30] demonstrates sensitivity of the dimension-two condensate to the Gribov horizon.

Moreover, one can readily argue that the ultraviolet divergent part of $<(A^a_\mu)^2>_{min}$ cannot be relevant to physical observables. Thus, we are left with the non-perturbative part of $<(A^a_\mu)^2>_{min}$ which is of order Λ^2_{QCD}. As we argued in [10] the monopoles realize

$$<|\phi_M|^2> \;\sim\; \Lambda^2_{QCD} \quad , \tag{22}$$

where ϕ_M is magnetically charged field. Properties of the branes seem to reflect the balance of locality and non-locality inherent to the gauge invariant operator of dimension 2 in gauge theories.

Acknowledgments

This note grew out of discussions of the results of the lattice measurements presented in [5,6,7,18,21]. I am thankful to V.G. Bornyakov, P.Yu. Boyko, M.N. Chernodub, F.V. Gubarev, A.V. Kovalenko, M.I. Polikarpov, T. Suzuki and S.N. Syritsyn for collaboration and thorough discussions.

I am thankful to A. Bartl, M. Faber, A. Di Giacomo, J. Greensite, J. Kubo, P. van Nieuwenhuizen, A.A. Slavnov, L. Stodolsky, H. Terao and E.T. Tomboulis for interesting discussions.

The paper was completed while the author was visiting the Yukawa Institute for Theoretical Physics of the Kyoto University. I am thankful to Prof. T. Kugo and other members of the group for the hospitality.

The material of this note was presented as a talk at the Conference "Confinement 2003" at RIKEN (Japan), July 21-24, (2003) and as a lecture at the Summer Institute, Fuji-Yoshida (Japan), August 12-19, (2003). I am thankful to the organizers, Prof. H. Suganuma and Prof. J. Kubo, for the invitations.

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